Transformation to Canonical Form

Consider $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with a, b, c, and d real numbers. Let λ_1 and λ_2 be the eignevalues of A with the corresponding eigenvectors V_1 and V_2, respectively.

1. λ_1 and λ_2 are real-valued and $\lambda_1 \neq \lambda_2$. Let T be the matrix whose columns are V_1 and V_2: $T = (V_1\ V_2)$. Then $T^{-1} A T = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$.

2. $\lambda = \lambda_1 = \lambda_2$.

   (a) V_1 and V_2 are linearly independent. In this case A is already in its canonical form: $A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$.

   (b) V_1 and V_2 are not linearly independent. Let V be V_1 or V_2 and let U be a solution of the matrix equation $(A - \lambda I)U = V$. Let T be the matrix whose columns are V and U: $T = (V\ U)$. Then $T^{-1} A T = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$.

3. λ_1 and λ_2 are complex-valued: $\lambda_1, \lambda_2 = \alpha \pm i\beta$ with $\beta \neq 0$. Let T be the matrix whose columns are $\operatorname{Re} V_1$ and $\operatorname{Im} V_1$: $T = (\operatorname{Re} V_1\ \operatorname{Im} V_1)$. Then $T^{-1} A T = \begin{pmatrix} \operatorname{Re} \lambda_1 & \operatorname{Im} \lambda_1 \\ -\operatorname{Im} \lambda_1 & \operatorname{Re} \lambda_1 \end{pmatrix}$.

Eigenvalues and Eigenvectors of Canonical 2×2 Matrices

Matrix	Eigenvalues	Eigenvectors
$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$	λ_1	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$
	λ_2	$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$
$\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$	λ	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$
$\begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$, $\beta \neq 0$	$\alpha + i\beta$	$\begin{pmatrix} 1 \\ i \end{pmatrix}$
	$\alpha - i\beta$	$\begin{pmatrix} 1 \\ -i \end{pmatrix}$